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Selberg Zeta Functions and the Dimensions of the Spaces of Elliptic Cusp Forms of Lower Weights

by

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§ 0. Introduction

0.1. Dimension formulas for certain spaces of elliptic cusp forms have been investigated by many authors. Among others Hiramatsu [Hil, 2 3], Hiramatsu-Akiyama [Hi-Aki], Tanigawa-Ishikawa [Ta-Ish], Akiyama [Aki] and Christian [Ch2, 3] studied the dimensions of the spaces of elliptic cusp forms of weight one in connection with certain zeta functions closely related to the Selberg trace formula. They represented the dimensions in terms of the residues at the poles of those zeta functions (In this case to represent the dimensions in a closed form with only arithmetic quantities seems to be impossible). To obtain the results the Selberg trace formula is an indispensable tool. The former four authors employed a formulation of the Selberg trace formula due originally to Selberg [Se] and to Kubota [Ku]. Christian [Ch2] used a formulation due to Fischer [Fi] which gives us a general exposition of the Selberg trace formula for $SL_2(\boldsymbol{R})$ via the Selberg zeta function. Though Christian discussed only the case of the principal congruence subgroup of $SL_2(\boldsymbol{Z})$, his method is applicable to any discrete subgroups of $SL_2(\boldsymbol{R})$ with finite covolume.

The aim of this paper is to give certain general dimension formulas of Hiramatsu's type for the spaces of elliptic cusp forms of lower weights along Christian's line ([Ch2]), and consequently to generalize some of the results of [Hi2, 3], [Hi-Aki], [Ta-Ishi], and [Ch2].

0.2. We summarize our results. Let Γ be a discrete subgroup of $SL_2(\boldsymbol{R})$ with finite covolume and χ a character of Γ (i.e., a unitary representation of Γ with dimension one). If $-1_2\in\Gamma$, we assume that $\chi(-1_2)=-1$. Denote by $\mathfrak{S}_1(\Gamma,\chi)$ the sapce of usual elliptic cusp forms of weight one with respect to Γ and χ. Hiramatsu [Hil, 2, 3] followed by Tanigawa-Ishikawa [Ta-Ish] and Christian [Ch2] defined a Selberg type zeta function $\zeta_{\Gamma,\chi}(s)$ as follows:

$$\zeta_{\Gamma,\chi}(s)=\delta_\Gamma\sum_{\{P\}_\Gamma}\frac{\operatorname{sgn}(\operatorname{tr}P)\chi(P)\log N(P_0)}{N(P)^{1/2}-N(P)^{-1/2}}\left(2\cosh\frac{\log N(P)}{2}\right)^{-2s},$$

where $\delta_\Gamma = 1/2$ or 1 according as $-1_2 \in \Gamma$ or not, and where $\{P\}_\Gamma$ denotes the Γ-conjugacy classes of hyperbolic elements of Γ and P_0 is the primitive hyperbolic element of Γ associated to P. The zeta function $\zeta_{\Gamma,\chi}(s)$ is continued analytically to a meromorphic function in the whole s-plane having a simple pole at $s=0$. Let $Z_{\Gamma,\chi}(s)$ be the Selberg zeta function associated with Γ and χ (for the definition, see (1.13) and (3.3)).

THEOREM 0.1. *Let χ be a $\{\pm 1\}$-valued character of Γ. Then,*

$$\dim \mathfrak{S}_1(\Gamma, \chi) = \frac{1}{2} \operatorname{Re} s_{s=0} \zeta_{\Gamma,\chi}(s) = \frac{1}{2} \operatorname{Re} s_{s=1/2} \left(\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s) \right).$$

Let $\Gamma_0(N)$ be the congruence subgroup of $SL_2(\boldsymbol{Z})$ given by

$$\Gamma_0(N) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, c \equiv 0 \bmod N \right\}.$$

THEOREM 0.2. *Let χ be a Dirichlet character* mod N *with* $\chi(-1) = -1$. *Let the same symbol χ denote the character of $\Gamma = \Gamma_0(N)$ defined by*

$$\chi(M) = \chi(d) \qquad \textit{for} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \tag{0.1}$$

Then,

$$\dim \mathfrak{S}_1(\Gamma_0(N), \chi) = \frac{1}{2} Re\, s_{s=0} \zeta_{\Gamma,\chi}(s) = \frac{1}{2} \operatorname{Re} s_{s=1/2} \left(\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s) \right).$$

Theorem 0.1 gives a generalization of [Hi-Aki, Theorem 2] and [Ch2, Satz 4, (248)]. Theorem 0.2 was first observed by Hiramatsu [Hi2, p. 185] and proved by Tanigawa-Ishikawa [Ta-Ish] if N is a prime integer.

In the following we obtain certain dimension formulas (Proposition 2.2, Theorem 2.5 in § 2) for the space $\mathfrak{S}_{2k}(\Gamma, \chi)$ of cusp forms of weight $2k$ $(1/2 \leq k \leq 1)$ with respect to Γ and χ, χ denoting a unitary multiplier system of Γ of weight $2k$. Theorem 0.1, Theorem 0.2 will be derived from those dimension formulas. To obtain our results we use Fischer's Selberg trace formula ([Fi]).

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NOTATION. Let $\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ denote the set of positive integers, the ring of rational integers, the real number field, and the complex number field, respectively. For any square matrix A with entries in $\boldsymbol{C}$, let ${}^tA$, $\operatorname{tr} A$, and $\det A$ denote the transposed matrix of A, the trace of A, and the determinant of A, respectively. We denote by 1_n the identity matrix of size n. For $z \in \boldsymbol{C}$, denote by $\operatorname{Re} z$ and $\operatorname{Im} z$ the real part of z and the imaginary part of z, respectively. For $x \in \boldsymbol{R}$, $x \neq 0$, let $\operatorname{sgn} x$ be the signature of x. Let $\Gamma(s)$ denote the gamma function. For a meromorphic function $f(z)$ having a pole at $z=\alpha$, we denote by $\operatorname{Res}_{z=\alpha}f(z)$ the residue of f at the pole $z=\alpha$. The symbol $e(\alpha)$ ($\alpha\in \boldsymbol{C}$) is used as an abbreviation for $\exp(2\pi i\alpha)$.

§ 1. Fischer's Selberg trace formula

We recall some basic results of Fischer [Fi].

Let $\mathfrak{H}$ be the upper half plane, on which the group $SL_2(\boldsymbol{R})$ acts in a usual manner. We write, for $g=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ of $SL_2(\boldsymbol{R})$ and $z\in\mathfrak{H}$,

$$gz=\frac{az+b}{cz+d} \quad \text{and} \quad J(g, z)=cz+d\,.$$

We put

$$d\omega(z)=y^{-2}dxdy \qquad \text{for} \quad x=\operatorname{Re} z\,, \quad y=\operatorname{Im} z\,,$$

which is an invariant volume element on $\mathfrak{H}$. In § 1 and § 2 we assume that Γ is a discrete subgroup of $SL_2(\boldsymbol{R})$ containing the element -1_2 and having a fundamental domain $\Gamma/\mathfrak{H}$ with $\operatorname{vol}(\Gamma/\mathfrak{H})$ finite, where

$$\operatorname{vol}(\Gamma/\mathfrak{H})=\int_{\Gamma\backslash\mathfrak{H}} d\omega(z)\,.$$

Let k be a fixed real number. Let $\sigma_{2k}(A, B)$ denote the cocycle given by

$$\sigma_{2k}(A, B)=e(2kw(A, B)) \quad (A, B)\in SL_2(\boldsymbol{R}))$$
$$2\pi w(A, B)=\arg J(A, Bz)+\arg J(B, z)-\arg J(AB, z) \qquad (z\in\mathfrak{H})\,,$$

where $\arg w$ ($w\in \boldsymbol{C}$-$\{0\}$) is chosen so that $-\pi<\arg w\leq\pi$, and where $w(A, B)$ is independent of the choice of z and takes the values $0, \pm 1$.

Let $(V, \langle\,,\rangle)$ be a d-dimensional $\boldsymbol{C}$-vector space equipped with a positive definite anti-hermitian scalar product $\langle\,,\rangle$. Namely, $\langle\,,\rangle$ is anti-linear in the first argument as in [Ro1, 2], [Fi]. Let $\mathscr{U}(V)$ be the group of unitary endomorphisms of V with respect to the scalar product $\langle\,,\rangle$.

A map $\chi: \Gamma\to\mathscr{U}(V)$ is called a unitary multiplier system of Γ of weight $2k$ and dimension d, if χ satisfies the following two properties

(1.1) a) $\chi(-1_2)=e^{-2\pi i k}\mathrm{id}_V$, id_V denoting the identity map of V,

b) $\chi(ST)=\sigma_{2k}(S, T)\chi(S)x(T) \qquad$ for all $\quad S, T\in\Gamma$.

The automorphic factor $j_S(z, k)$ is defined by

$$j_S(z, k)=\exp(2ik \arg J(S,z)) \qquad \text{for} \quad S\in SL_2(\boldsymbol{R}) \quad \text{and} \quad z\in\mathfrak{H}\,.$$

We write $j_S(z)$ for $j_S(z, k)$ if the there is no fear of confusion. Then,

(1.2) $$j_S(Tz)j_T(z)=\sigma_{2k}(S, T)j_{ST}(z) \qquad \text{for} \quad S, T\in SL_2(\boldsymbol{R}) \text{ and } z\in\mathfrak{H}\,.$$

Let $\mathscr{H}_k$ be the space of measurable functions $f: \mathfrak{H}\to V$ satisfying

a) $f(Mz)=\chi(M)j_M(z)f(z)$ for all $M\in\Gamma$,

b) $\int_{\Gamma\backslash\mathfrak{H}}\langle f(z), f(z)\rangle d\omega(z)<+\infty$.

The Petersson scalar product (f, g) on $\mathscr{H}_k$ is defined by

$$(f, g)=\int_{\Gamma\backslash\mathfrak{H}}\langle f(z), g(z)\rangle d\omega(z) \qquad (f, g\in\mathscr{H}_k).$$

Then $\mathscr{H}_k$ forms a Hilbert space via this scalar product. We put

$$\Delta_k=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-2iky\frac{\partial}{\partial x}.$$

Denote by $\mathscr{D}_k$ the set of all twice continuously differentiable functions $f\in\mathscr{H}_k$ with $\Delta_k f\in\mathscr{H}_k$. It is shown by Roelcke that

$$(-\Delta_k f, g)=(f, -\Delta_k g) \qquad \text{for} \quad f, g\in\mathscr{D}_k \qquad \text{([Ro1, Satz 3.1])}$$

and moreover that the linear operator $-\Delta_k: \mathscr{D}_k\to\mathscr{H}_k$ has the unique self-adjoint extension $-\Delta_k^{\sim}: \mathscr{D}_k^{\sim}\to\mathscr{H}_k$ which is a closed operator ([Ro1, Satz 3.2]).

We assume that Γ has parabolic elements. Otherwise the situation is easier. For each cusp ζ of Γ, denote by Γ_ζ the stabilizer of ζ in Γ: $\Gamma_\zeta=\{M\in\Gamma\mid M\zeta=\zeta\}$. Let $\zeta_1, \cdots, \zeta_\tau$ be a complete system of representatives of the Γ-equivalent classes of cusps of Γ. Set

$$U=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\in SL_2(\boldsymbol{R}).$$

One can choose $A_1, \cdots, A_\tau\in SL_2(\boldsymbol{R})$ such that -1_2 and $T_j=A_j^{-1}UA_j$ generate the group Γ_{ζ_j}. Let m_j denote the multiplicity of the eigen value 1 of $\chi(T_j)$. For each j $(1\le j\le\tau)$ there exists an orthonormal basis $\{v_{j1}, \cdots, v_{jd}\}$ of V with the property

$$\chi(T_j)v_{jp}=e(\beta_{jp})v_{jp}, \qquad \text{where} \quad \begin{cases}\beta_{jp}=0 & \text{if} \quad 1\le p\le m_j,\\ 0<\beta_{jp}<1 & \text{if} \quad m_j+1\le p\le d.\end{cases} \tag{1.3}$$

We set

$$\tau^*=\tau^*(\Gamma)=\sum_{j=1}^{\tau} m_j. \tag{1.4}$$

Then, τ^* is independent of the choice of the system $\zeta_1, \cdots, \zeta_\tau$ ([Fi, Remark 1.5.2]). Let j, p be positive integers with $1\le j\le\tau$, $1\le p\le m_j$. Roelcke [Ro2, § 10] studied the Eisenstein series $E_{jp}(z, s)$ in detail:

$$E_{jp}(z, s)=\sum_{M\in\Gamma_{\zeta_j}\backslash\Gamma}\sigma_{2k}(A_j, M)^{-1}\chi(M)^{-1}v_{jp}j_{A_jM}(z)^{-1}(\operatorname{Im} A_jMz)^s \qquad (z\in\mathfrak{H}).$$

The Eisenstein series $E_{jp}(z, s)$ is absolutely convergent for $\operatorname{Re} s>1$. For the Fourier expansion of $E_{jp}(z, s)$, we refer to [Ro1, pp. 301, 302], [Ro2, p. 294, Lemma 10.2] and [Fi, Propositions 1.5.4, 1.5.6]. The Eisenstein series $E_{jp}(z, s)$ has a Fourier expansion at a cusp ζ_l $(1\leq l\leq\tau)$ of the form

$$E_{jp}(z, s)=j_{A_l}(z)^{-1}(\mathfrak{u}_{jp,l}(\operatorname{Im} A_l z)+\mathfrak{q}_{jp,l}(A_l z)),$$

where $\mathfrak{u}_{jp,l}(y)$, $\mathfrak{q}_{jp,l}(z)$ are given as follows:

$$(1.5)\qquad u_{jp,l}(y)=\delta_{jl}y^s v_{jp}+\mathfrak{p}_{jp,l}(s)y^{1-s}\qquad (y>0),$$

$$\mathfrak{q}_{jp,l}(z)=\sum_{r=1}^{m_l}\left\{\sum_{n\in \boldsymbol{Z}-\{0\}} c_{jp,lr}^{(n)}(y)e(nx)\right\}v_{lr}$$

$$+\sum_{r=m_l+1}^{d}\left\{\sum_{n\in \boldsymbol{Z}} c_{jp,lr}^{(n)}(y)e((n+\beta_{lr})x)\right\}v_{lr}\qquad (z\in\mathfrak{H})$$

with certain $\mathfrak{p}_{jp,l}(s)\in V$ and Fourier coefficients $c_{jp,lr}^{(n)}(y)\in \boldsymbol{C}$. Furthermore, $\mathfrak{p}_{jp,l}(s)$ has the expression

$$(1.6)\qquad \mathfrak{p}_{jp,l}(s)=\sum_{r=1}^{m_l}\varphi_{jp,lr}(s)v_{lr}\qquad \text{with some}\quad \varphi_{jp,lr}(s)\in \boldsymbol{C}.$$

The so-called scattering matrix $\Phi(s)$ of size τ^* is defined by

$$(1.7)\qquad \Phi(s)=(\varphi_{jp,lr}(s))_{j,l=1,\cdots,\tau;\,p=1,\cdots,m_j;\,r=1,\cdots,m_l},$$

where jp is the line index, lr the column index both in lexicographic order. It is known by [Ro2, (10.30)] that

$$(1.8)\qquad {}^t\overline{\Phi(\bar{s})}=\Phi(s)\qquad (\text{i.e., } \overline{\varphi_{jp,lr}(\bar{s})}=\varphi_{lr,jp}(s)).$$

We define $\boldsymbol{E}(z, s)$ by lining up $E_{jp}(z, s)$ as a column vector in the lexicographic order:

$$\boldsymbol{E}(z, s)=(E_{jp}(z, s))_{j=1,\cdots,\tau;\,p=1,\cdots,m_j}.$$

The main result concerning the Eisenstein series $E_{jp}(z, s)$ is the following due to Roelcke [Ro2, Satz 10.2] (see also [Fi, pp. 31–34]).

THEOREM (Roelcke). *The Eisenstein series $E_{jp}(z, s)$ and $\varphi_{jp,lr}(s)$ are continued analytically to meromorphic functions in the whole s-plane which are holomorphic on the critical line* $\operatorname{Re} s=1/2$. *Moreover they satisfy the functional equations*

$$\boldsymbol{E}(z, 1-s)=\Phi(1-s)\boldsymbol{E}(z, s)\quad \textit{and}\quad \Phi(s)\Phi(1-s)=1_{\tau^*}.$$

For $M=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ of $SL_2(\boldsymbol{R})$, we write $c(M)$ for c. For each pair (j, l) $(1\leq j, l\leq\tau)$, denote by $B_{jl}^+(\Gamma)$ the subset of Γ consisting of $M\in\Gamma$ with $c(A_jMA_l^{-1})>0$. Denote by $\Gamma_{\zeta_j}^+$ the subgroup of Γ_{ζ_j} generated by T_j. Then,

$$\Gamma_{\zeta_j}^+=A_j^{-1}\Gamma_0A_j,\qquad \Gamma_0 \text{ being the subgroup } \left\{\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\middle|\, m\in \boldsymbol{Z}\right\}.$$

The following lemma is known in the literature ([Hej, p. 368 (5.22)]). Only the case of $2k$ being an integer is discussed for the later use.

LEMMA 1.1. *Assume that $2k$ is an integer. Let j, l, p, r be positive integers with $1 \leq j, l \leq \tau$, $1 \leq p \leq m_j$, $1 \leq r \leq m_l$. Then,*

$$\varphi_{jp,lr}(s) = e^{-\pi i k}\gamma(s, k) \sum_{M \in \Gamma_{\zeta_j}^+ \backslash B_{jl}^+(\Gamma)/\Gamma_{\zeta_l}^+} \frac{\langle \chi(M)v_{lr}, v_{jp}\rangle}{c(A_j M A_l^{-1})^{2s}},$$

where we set

$$\gamma(s, k) = \frac{2^{2-2s}\pi\Gamma(2s)}{\Gamma(s+k)\Gamma(s-k)}.$$

The above infinite series is absolutely convergent for $\operatorname{Re} s > 1$.

Proof. We note that there exists $M \in \Gamma$ with $c(A_j M A_l^{-1}) = 0$ if and only if $j = l$. By the assumption for k, $\sigma_{2k}(A, B) = 1$ for all $A, B \in SL_2(\boldsymbol{R})$. Thus we have, by the definition of $E_{jp}(z, s)$,

$$\langle v_{lr}, j_{A_l}(A_l^{-1}z)E_{jp}(A_l^{-1}z, s)\rangle = \delta_{jl}y^s\langle v_{lr}, v_{jp}\rangle \tag{1.9}$$

$$+ \sum_{M \in \Gamma_{\zeta_j}^+ \backslash B_{jl}^+(\Gamma)} j_{A_l}(A_l^{-1}z) j_{A_j M}(A_l^{-1}z)^{-1}(\operatorname{Im} A_j M A_l^{-1}z)^s \langle v_{lr}, \chi(M)^{-1}v_{jp}\rangle .$$

We write each element of $\Gamma_{\zeta_j}^+ \backslash B_{jl}^+(\Gamma)$ in the form MM' with $M \in \Gamma_{\zeta_j}^+ \backslash B_{jl}^+(\Gamma)/\Gamma_{\zeta_l}^+$, $M' \in \Gamma_{\zeta_l}^+$. Each M' has an expression $M' = A_l^{-1}U^n A_l$ with $n \in \boldsymbol{Z}$. Using the property b) of (1.1) and the relation (1.2), we have

$$j_{A_l}(A_l^{-1}z) j_{A_j MM'}(A_l^{-1}z)^{-1}\chi(MM')^{-1} = j_{A_j M A_l^{-1}}(z+n)^{-1}\chi(M')^{-1}\chi(M)^{-1} .$$

Since $\chi(M')v_{lr} = v_{lr}$ $(1 \leq r \leq m_l)$, the right side of (1.9) coincides with

$$\delta_{jl}y^s\langle v_{lr}, v_{jp}\rangle + \sum_{M \in \Gamma_{\zeta_j}^+ \backslash B_{jl}^+(\Gamma)/\Gamma_{\zeta_l}^+} \sum_{n \in \boldsymbol{Z}}$$

$$j_{A_j M A_l^{-1}}(z+n)^{-1}(\operatorname{Im} A_j M A_l^{-1}(z+n))^s \langle \chi(M)v_{lr}, v_{jp}\rangle ,$$

which is invariant under the transformation $z \to z+1$. Thus,

$$\int_0^1 \langle v_{lr}, j_{A_l}(A_l^{-1}z)E_{jp}(A_l^{-1}z, s)\rangle dx = \delta_{jl}y^s\langle v_{lr}, v_{jp}\rangle \tag{1.10}$$

$$+ \sum_{M \in \Gamma_{\zeta_j}^+ \backslash B_{jl}^+(\Gamma)/\Gamma_{\zeta_j}^+} \langle \chi(M)v_{lr}, v_{jp}\rangle \cdot \int_{-\infty}^{\infty} j_{A_j M A_l^{-1}}(z)^{-1}(\operatorname{Im} A_j M A_l^{-1}z)^s dx .$$

If we write $A_j M A_l^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c > 0$, the last integral equals

$$\frac{y^s}{c^{2s}} \int_{-\infty}^{\infty} \frac{1}{(z+c^{-1}d)^{2k}|z+c^{-1}d|^{2s-2k}} dx \qquad (x = \operatorname{Re} z, y = \operatorname{Im} z) .$$

An easy calculation shows that this value coincides with

$$\frac{y^{1-s}}{c^{2s}} \cdot e^{-\pi i k} \gamma(s, k) .$$

Since the scalar product $\langle v_{lr}, u_{jp,l}(y) \rangle$ is equal to the right side of (1.10), the identity in Lemma 1.1 can be derived from (1.5), (1.6), and (1.10). The absolutely convergence for $\mathrm{Re}\, s > 1$ of the infinite series follows from that of $E_{jp}(z, s)$ in the same region.

q.e.d.

Let $\psi(s) = \Gamma'(s)/\Gamma(s)$. We set

$$\xi_I(s) = -\frac{d\mathrm{vol}(\Gamma \backslash \mathfrak{H})}{4\pi} \cdot (2s-1)(\psi(s+k) + \psi(s-k)) . \tag{1.11}$$

Every hyperbolic element P of Γ has an expression

$$P = \pm A \begin{pmatrix} N(P)^{1/2} & 0 \\ 0 & N(P)^{-1/2} \end{pmatrix} A^{-1}, \ N(P) > 1 \qquad \text{with some} \quad A \in SL_2(\boldsymbol{R}) . \tag{1.12}$$

The uniquely determined number $N(P)$ is called the norm of P. A hyperbolic element P_0 of Γ is called primitive, if P_0 with the element -1_2 generates the centralizer $Z_\Gamma(P_0)$ of P_0 in Γ. For each hyperbolic element P of Γ, there exists the unique primitive hyperbolic element P_0 of Γ such that $P = P_0^m$ with some $m \in N$ and $\mathrm{sgn}(\mathrm{tr}\, P) = \mathrm{sgn}(\mathrm{tr}\, P_0)$. Then P_0 is called the primitive hyperbolic element associated to P. Denote by $\{P\}_\Gamma$ (resp. $\{P_0\}_\Gamma$) the Γ-conjugacy classes of hyperbolic (resp. primitive hyperbolic) elements of Γ. The Selberg zeta function $Z_{\Gamma,\chi}(s)$ associated with Γ, χ is defined by

$$Z_{\Gamma,\chi}(s) = \prod_{\{P_0\}_\Gamma \, \mathrm{tr}\, P_0 > 2} \ \prod_{m=0}^{\infty} \det(\mathrm{id}_V - \chi(P_0) N(P_0)^{-s-m}) . \tag{1.13}$$

It is known that $Z_{\Gamma,\chi}(s)$ is absolutely convergent for $Re\, s > 1$ and indicates a holomorphic function in the same region ([Fi, Corollary 2.2.6]). Fischer proved that $Z_{\Gamma,\chi}(s)$ has a meromorphic continuation to the whole s-plane ([Fi, p. 116, (3.1.4)]. We define $\xi_{hyp}(s)$ to be the logarithmic derivative of $Z_{\Gamma,\chi}(s)$:

$$\xi_{hyp}(s) = (Z'_{\Gamma,\chi}/Z_{\Gamma,\chi})(s) .$$

Then, $\xi_{hyp}(s)$ has the following expression

$$\xi_{hyp}(s) = \sum_{\{P\}_\Gamma \, \mathrm{tr}\, P > 2} \mathrm{tr}\, \chi(P) \log N(P_0) \cdot \frac{N(P)^{-s}}{1 - N(P)^{-1}} , \tag{1.14}$$

where P_0 is the primitive hyperbolic element of Γ associated to P. The infinite series on the right side of (1.14) is absolutely convergent for $\mathrm{Re}\, s > 1$ and $\xi_{hyp}(s)$ indicates a holomorphic function in $\mathrm{Re}\, s > 1$ ([Fi, (2.2.1)]). We set

$$(1.15)\qquad \zeta_{\Gamma,\chi}(s)=\sum_{\{P\}_\Gamma \operatorname{tr} P>2}\frac{\operatorname{tr}\chi(P)\log N(P_0)}{N(P)^{1/2}-N(P)^{-1/2}}\left(2\cosh\frac{\log N(P)}{2}\right)^{-2s}.$$

Since $\cosh\frac{1}{2}\log x\sim x^{1/2}$ as $x\to+\infty$, the absolutely convergence for $\operatorname{Re}s>1/2$ of $\zeta_{\Gamma,\chi}(s)$ is similarly verified with the help of Lemma 2.2.2 in [Fi]. Simultaneously, $\zeta_{\Gamma,\chi}(s)$ indicates a holomorphic function in the half plane $\operatorname{Re}s>1/2$ (see [Ch2, Satz 3]). We note that $Z_{\Gamma,\chi}(s)$, $\zeta_{\Gamma,\chi}(s)$ depend on k mod $\boldsymbol{Z}$, since χ defines a multiplier system of Γ of weight $2k$ if and only if χ is also that of weight $2(k+l)$ for any $l\in\boldsymbol{Z}$ ([Fi, Remark 1.3.5]).

Every elliptic element R of Γ is conjugate in $SL_2(\boldsymbol{R})$ to some element $\left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right)$ with $0<\theta<2\pi$, $\theta\neq\pi$, where the argument θ is uniquely determined. In the case to be specified we write $\theta(R)$ for θ. We write $2\nu(R)$ for the order of the centralizer $Z_\Gamma(R)$ in Γ. There exists the unique elliptic element $R_0\in Z_\Gamma(R)$ which is conjugate to $\left(\begin{smallmatrix}\cos\pi/\nu & -\sin\pi/\nu\\ \sin\pi/\nu & \cos\pi/\nu\end{smallmatrix}\right)$ in $SL_2(\boldsymbol{R})$ with ν denoting $\nu(R)$. The unique R_0 is called the primitive elliptic element corresponding to R. Every elliptic element S of $Z_\Gamma(R)$ has the unique expression

$$S=R_0^n\qquad\text{with}\quad n\in\{1,\cdots,\nu-1,\nu+1,\cdots,2\nu-1\}.$$

We set

$$(1.16)\qquad \xi_{ell}(s)=\sum_{\{R\}_\Gamma\, 0<\theta<\pi}\operatorname{tr}\chi(R)\frac{ie^{2ik\theta}}{2\nu(R)^2\sin\theta}\times\sum_{l=0}^{\nu(R)-1}\left(e^{i\theta(2l+1)}\psi\left(\frac{s+k+l}{\nu(R)}\right)-e^{-i\theta(2l+1)}\psi\left(\frac{s-k+l}{\nu(R)}\right)\right),$$

where the first summation is taken over the Γ-conjugacy classes $\{R\}_\Gamma$ of elliptic elements of Γ with $0<\theta(R)<\pi$.

Let the numbers $\{\beta_{jp}\}_{j=1,\cdots,\tau;p=1,\cdots,d}$, $\{m_j\}_{j=1,\cdots,\tau}$, τ^* be the same as in (1.3), (1.4). We set

$$\begin{aligned}(1.17)\qquad \xi_{par}(s)=&-d\tau\log 2-\log\prod_{j=1}^{\tau}\prod_{p=m_j+1}^{d}\sin\pi\beta_{jp}\\&+(\psi(s+k)-\psi(s-k))\left(\frac{1}{2}d\tau-\sum_{j=1}^{\tau}\sum_{p=1}^{d}\beta_{jp}\right)\\&+\tau^*\Big(\psi(s-k)-\psi(s)-\psi(s+1/2)\Big)+\frac{1}{2s-1}\operatorname{tr}(1_{\tau^*}-\Phi(1/2))\\&+\frac{(2s-1)}{4\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(s-1/2)^2+t^2}-\frac{1}{1/4+t^2}\right)\frac{\varphi'}{\varphi}(1/2+it)dt,\end{aligned}$$

where

$$\varphi(s)=\det\Phi(s).$$

As is shown in [Fi, Lemma 2.4.19], the last integral on the right side of the equality (1.17) is aboslutely convergent for $\operatorname{Re} s > 1/2$ and indicates a holomorphic function of s in that domain.

In view of the results of Roelcke ([Ro1, 2, Satz 5.7, Satz 7.2]) there exists a complete set of the eigen values $\lambda_0, \lambda_1, \cdots, \lambda_n, \cdots$ counted with multiplicites of the self-adjoint operator $-\Delta_k^{\sim}: \mathscr{D}_k^{\sim} \to \mathscr{H}_k$. One can write $\lambda_n = 1/4 + r_n^2$ with $r_n \in \{x \mid x \geq 0\} \cup \{iy \mid y > 0\}$. We set

$$(1.18) \qquad S_{\Gamma,\chi}(s, a) = \sum_{n=0}^{\infty} \left(\frac{1}{(s-1/2)^2 + r_n^2} - \frac{1}{(a-1/2)^2 + r_n^2} \right) \qquad \text{for} \quad s, a \in \boldsymbol{C} - P,$$

P denoting the subset $\{1/2 \pm ir_n \mid n = 0, 1, \cdots\}$ of $\boldsymbol{C}$. As is easily seen from [Fi, Theorem 1.6.5], the infinite series $S_{\Gamma,\chi}(s, a)$ is absolutely convergent for any $s, a \in \boldsymbol{C} - P$. Moreover, $S_{\Gamma,\chi}(s, a)$ with a being fixed indicates a holomorphic function of s in the domain $\boldsymbol{C} - P$.

Now we can formulate Fischer's resolvent trace formula.

THEOREM (Fischer [Fi, Theorem 2.5.2]). *Let χ be a multiplier system of Γ of weight $2k \in \boldsymbol{R}$ and dimension d. Assume that s, $a \in \boldsymbol{C}$ with* $\operatorname{Re} s$, $\operatorname{Re} a > 1$, $|k| - s$, $|k| - a \notin \boldsymbol{N} \cup \{0\}$. *Then,*

$$(1.19) \qquad S_{\Gamma,\chi}(s, a) = \frac{1}{2s-1} \xi(s) - \frac{1}{2a-1} \xi(a),$$

where we set

$$\xi(s) = \xi_I(s) + \xi_{hyp}(s) + \xi_{ell}(s) + \xi_{par}(s).$$

Fischer derived the general Selberg trace formula ([Fi, Theorem 4.1.1]) from his resolvent trace formula. Set, for simplicity,

$$k^* = \max(1/2, |k| - 1/2).$$

Let $\delta > 0$. We choose as test functions of the general Selberg tace formula holomorphic functions $h(r)$ satisfying

$$(1.20) \qquad \begin{cases} \text{(i)} & h: \{r \in \boldsymbol{C} \mid |\operatorname{Im} r| < k^* + \delta\} \to \boldsymbol{C}, \\ \text{(ii)} & h(r) = h(-r) \qquad \text{for all} \quad r, \\ \text{(iii)} & |h(r)| = O(|\operatorname{Re} r|^{-4-\delta}) \qquad \text{as} \quad |\operatorname{Re} r| \to +\infty. \end{cases}$$

The Fourier transform of h is defined by

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iru} dr.$$

Then the inverse Fourier transform is given by

$$h(r)=\int_{-\infty}^{\infty} g(u)e^{iru}du\,.$$

The following is a prototype of the general Selberg trace formula.

THEOREM (Fischer [Fi, Lemma 4.1.3]). *Take $\delta>0$ and a number B with $k^*<B<k^*+\delta$. Let $h(r)$ be a holomorphic function satisfying the condition* (1.20). *Then the series $\sum_{n=0}^{\infty} h(r_n)$ converges absolutely, and*

$$\sum_{n=0}^{\infty} h(r_n)=\frac{1}{2\pi}\int_{-\infty-iB}^{\infty-iB} h(\zeta)(1/2+i\zeta)d\zeta\,. \tag{1.21}$$

Following Christian [Ch2], we modify the right side of (1.21) in a manner slightly different from [Fi, pp. 170–175]. We get another version of the general Selberg trace formula.

PROPOSITION 1.2. *Assume that $k\geq 0$. Take $\delta>0$. Let $h(r)$ and $g(u)$ be as above. Set $N_0=N\cup\{0\}$. Then,*

$$\begin{aligned}
\sum_{n=0}^{\infty} h(r_n)=&\frac{d\,\mathrm{vol}(\Gamma\diagdown\mathfrak{H})}{2\pi}\sum_{l<k-1/2,\,l\in N_0}(k-1/2-l)h(i(k-1/2-l))\\
&+\sum_{\{P\}_\Gamma \mathrm{tr}\,\chi P>2}\mathrm{tr}\,\chi(P)\cdot\frac{\log N(P_0)}{N(P)^{1/2}-N(P)^{-1/2}}g(\log N(P))\\
&+\sum_{\substack{\{R\}_\Gamma\\ 0<\theta<\pi}}\ \sum_{\substack{l\leq k-1/2\\ l\in N_0}}\frac{\mathrm{tr}\,\chi(R)\cdot ie^{i\theta}(2k-1-2l)}{2\nu(R)\sin\theta}\cdot\varepsilon_{k,l}h(i(k-1/2-l))\\
&+\left(\frac{d\tau}{2}-\sum_{j=1}^{\tau}\sum_{p=1}^{d}\beta_{jp}\right)\cdot\sum_{l\leq k-1/2,\,l\in N_0}\varepsilon_{k,l}h(i(k-1/2-l))\\
&-\tau^*\sum_{l\leq k-1/2,\,l\in N_0}\varepsilon_{k,l}h(i(k-1/2-l))+\frac{1}{4}h(0)\mathrm{tr}(1_{\tau^*}-\Phi(1/2))\\
&+\frac{1}{4\pi}\int_{-\infty}^{\infty}h(r)\frac{\varphi'}{\varphi}(1/2+ir)dr+\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)f_1(r)dr\,,
\end{aligned}\tag{1.22}$$

where

$$\varepsilon_{k,l}=\begin{cases}1\cdots & l<k-1/2\\ 1/2\cdots & l=k-1/2\end{cases}$$

and $f_1(\zeta)$ is a certain meromorphic function in the whole complex plane having only simple poles and satisfying the estimate on the real line:

$$f_1(r)=O(|r|^{1+\varepsilon})\qquad (r\in\boldsymbol{R},\ |r|\to+\infty)$$

with any $\varepsilon>0$. The explicit form of f_1 is given in (1.23) *below.*

Proof. First we note that $\psi(z)$ has simple poles at $z=-n$ ($n\in N_0$) with residues -1. We begin with the identity (1.21). The following two integrals have been calculated in [Fi, Lemma 4.1.4, Lemma 4.1.5]:

$$\frac{1}{2\pi}\int_{-\infty-iB}^{\infty-iB} h(\zeta)\xi_I(1/2+i\zeta)d\zeta=\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)f_2(r)dr$$

$$+\frac{d\operatorname{vol}(\Gamma\backslash\mathfrak{H})}{2\pi}\sum_{l<k-1/2,\,l\in N_0}(k-1/2-l)h(i(k-1/2-l))$$

with

$$f_2(\zeta)=\frac{d\operatorname{vol}(\Gamma\backslash\mathfrak{H})}{2}\cdot\frac{\zeta\sinh 2\pi\zeta}{\cosh 2\pi\zeta+\cos 2\pi k},$$

and

$$\frac{1}{2\pi}\int_{-\infty-iB}^{\infty-iB} h(\zeta)\xi_{hyp}(1/2+i\zeta)d\zeta=\sum_{\substack{\{P\}_\Gamma\\ \operatorname{tr}P>2}}\operatorname{tr}\chi(P)\frac{\log N(P_0)}{N(P)^{1/2}-N(P)^{-1/2}}g(\log N(P)),$$

where the last infinite series converges absolutely. Wet set

$$f_3(\zeta)=\xi_{ell}(1/2+i\zeta)-\delta^*_{k,1/2}\sum_{\{R\}_\Gamma 0<\theta<\pi}\frac{\operatorname{tr}\chi(R)}{2\nu(R)\sin\theta}\cdot\frac{1}{\zeta},$$

where

$$\delta^*_{k,1/2}=\begin{pmatrix}0\cdots k\equiv 1/2 \bmod Z\\ 1\cdots k\equiv 1/2 \bmod Z.\end{pmatrix}$$

We note that $f_3(\zeta)$ is holomorphic at $\zeta=0$. Using the formula

$$\int_{-\infty-iB}^{\infty-iB} h(\zeta)\frac{1}{\zeta}d\zeta=\pi i h(0),$$

We see immediately that

$$\frac{1}{2\pi}\int_{-\infty-iB}^{\infty-iB} h(\zeta)\xi_{ell}(1/2+i\zeta)d\zeta=\frac{1}{2\pi}\int_{-\infty-iB}^{\infty-iB} h(\zeta)f_3(\zeta)d\zeta$$

$$+\delta^*_{k,1/2}\cdot\frac{i}{2}\cdot\sum_{\{R\}_\Gamma 0<\theta<\pi}\frac{\operatorname{tr}\chi(R)}{2\nu(R)\sin\theta}\cdot h(0).$$

The residue theorem enables us to shift the path of integration to the real axis. Calculating the residues at the poles of the function $h(\zeta)f_3(\zeta)$ in the horizontal strip $\{\zeta\in C\mid -B<\operatorname{Im}\zeta<0\}$, we get

$$\frac{1}{2\pi}\int_{-\infty-iB}^{\infty-iB} h(\zeta)\xi_{ell}(1/2+i\zeta)d\zeta=\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)f_3(r)dr$$

$$+\sum_{\substack{\{R\}_\Gamma\\ 0<\theta<\pi}}\sum_{\substack{l\leq k-1/2\\ l\in N_0}}\frac{\operatorname{tr}\chi(R)\cdot ie^{i\theta(2k-1-2l)}}{2\nu(R)\sin\theta}\cdot\varepsilon_{k,l}h(i(k-1/2-l))\,.$$

In a manner similar to the proof of [Fi, Lemma 4.1.7], we have

$$\frac{1}{2\pi}\int_{-\infty-iB}^{\infty-iB} h(\zeta)\xi_{par}(1/2+i\zeta)d\zeta=g(0)\left(-d\tau\log 2-\log\prod_{j=1}^{\tau}\prod_{p=m_j+1}^{d}\sin\pi\beta_{jp}\right)$$

$$+\int_1\cdot\left(\frac{d\tau}{2}-\sum_{j=1}^{\tau}\sum_{p=1}^{d}\beta_{jp}\right)+\frac{1}{4}h(0)\operatorname{tr}(1_{\tau*}-\Phi(1/2))+\tau^*\int_2+\int_3\,,$$

where we set

$$\int_1=\frac{1}{2\pi}\int_{-\infty-iB}^{\infty-iB} h(\zeta)(\psi(1/2+k+i\zeta)-\psi(1/2-k+i\zeta))d\zeta\,,$$

$$\int_2=\frac{1}{2\pi}\int_{-\infty-iB}^{\infty-iB} h(\zeta)(\psi(1/2-k+i\zeta)-\psi(1/2+i\zeta)-\psi(1+i\zeta))d\zeta\,,$$

$$\int_3=\frac{1}{2\pi}\int_{-\infty-iB}^{\infty-iB} h(\zeta)\frac{1}{4\pi i}\int_{-\infty}^{\infty}\left[\frac{1}{\zeta+t}+\frac{1}{\zeta-t}+2\zeta\left(\frac{1}{1/2+it}+\frac{1}{1/2-it}\right)\right]\cdot\frac{\varphi'}{\varphi}(1/2+it)dtdr\,.$$

The residue theorem again implies that

$$\int_1=\sum_{l\leq k-/2,\, l\in N_0}\varepsilon_{k,l}h(i(k-1/2-l))+\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)f_4(r)dr\,,$$

$$\int_2=-\sum_{l\leq k-1/2,\, l\in N_0}\varepsilon_{k,l}h(i(k-1/2-l))+\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)f_5(r)dr\,,$$

where

$$f_4(\zeta)=\psi(1/2+k+i\zeta)-\left(\psi(1/2-k+i\zeta)+\delta^*_{k,1/2}\cdot\frac{1}{i\zeta}\right),$$

$$f_5(\zeta)=\psi(1/2-k+i\zeta)+\delta^*_{k,1/2}\cdot\frac{1}{i\zeta}-\psi(1/2+i\zeta)-\psi(1+i\zeta)\,.$$

It has been proved in [Fi, pp. 173–175] that the integral $\int_3$ absolutely converges and equals

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{\varphi'}{\varphi}(1/2+ir)dr\,.$$

Now we set

$$f_1(\zeta)=f_2(\zeta)+f_3(\zeta)+\left(-d\tau\log 2-\log\prod_{j=1}^{\tau}\prod_{p=m_j+1}^{d}\sin\pi\beta_{jp}\right) \tag{1.23}$$
$$+\left(\frac{d\tau}{2}-\sum_{j=1}^{\tau}\sum_{p=1}^{d}\beta_{jp}\right)f_4(\tau)+\tau^*f_5(\zeta).$$

The function $\psi(z)$ satisfies the estimate

$$\psi(z)=\log z-\frac{1}{2z}+O(|z|^{-2}) \qquad \text{as} \quad |z|\to+\infty, \quad |\arg z|\leq\pi-\lambda,$$

λ being a fixed positive number with $\lambda<\pi$ (see [Fi, (3.2.2)]). Thus it follows from the definition of $f_j(\zeta)$ $(2\leq j\leq 5)$ that, for any $\varepsilon>0$,

$$f_1(r)=O(|r|^{1+\varepsilon}) \qquad (r\in\boldsymbol{R}, |r|\to+\infty).$$

Moreover, $f_1(\zeta)$ is a meromorphic function in the whole ζ-plane with only simple poles. Thus we obtain Proposition 1.2. q.e.d.

§ 2. The dimensions of the spaces of elliptic cusp forms; the cases of $-1_2\in\Gamma$

We keep the notation used in § 1. We define an automorphic factor $J(M, z)^{2k}$ by

$$J(M, z)^{2k}=|J(M, z)|^{2k}j_M(z) \qquad M\in SL_2(\boldsymbol{R}), \qquad z\in\mathfrak{H},$$

where $j_M(z)=j_M(z, k)$. Let (χ, V) be a unitary multiplier system of Γ of weight $2k$ and dimension d. A holomorphic function $f: \mathfrak{H}\to V$ is called a modular form of weight $2k$ with respect to Γ and χ, if f satisfies

(i) $f(Mz)=\chi(M)J(M, z)^{2k}f(z)$ for all $M\in\Gamma$,

(ii) if $A\infty$ $(A\in SL_2(\boldsymbol{R}))$ is a cusp of Γ, there exists $\varepsilon>0$ such that $|J(A, z)|^{-2k}|f(Az)|$ is bounded on the domain $\{z\in\mathfrak{H}\,|\,\mathrm{Im}\,z>\varepsilon\}$.

The space of modular forms of weight $2k$ with respect to Γ and χ is denoted by $\mathfrak{M}_{2k}(\Gamma, \chi)$ (for which Fischer used the notation $\{\Gamma, -2k, \chi\}$ in [Fi, p. 110]). Let $Q(\Gamma, -2k, \chi)$ denote the subspace of $\mathfrak{M}_{2k}(\Gamma, \chi)$ consisting of $f\in\mathfrak{M}_{2k}(\Gamma, \chi)$ with

$$\int_{\Gamma\backslash\mathfrak{H}}\langle f(z), f(z)\rangle y^{2k}d\omega(z)<+\infty.$$

An element f of $\mathfrak{M}_{2k}(\Gamma, \chi)$ is called a cusp form of weight $2k$ with respect to Γ and χ, if for any $A\in SL_2(\boldsymbol{R})$ such that $A\infty$ is a cusp of Γ, there exists $\varepsilon>0$ with the estimate

$$|J(A, z)|^{-2k}|f(Az)|=O(e^{-\varepsilon y}) \qquad \text{as} \quad y\to+\infty \quad (y=\mathrm{Im}\,z).$$

Let $\mathfrak{S}_{2k}(\Gamma, \chi)$ denote the space of those cusp forms.

For $s\in\boldsymbol{C}$, denote by $\mathscr{H}_k(s)$ the subspace of $\mathscr{H}_k$ consisting of $f\in\mathscr{D}_k^{\sim}$ with $-\Delta_k^{\sim}f=s(1-s)f$. It follows from the definition of r_n that $\mathscr{H}_k(s)\neq\{0\}$ if and only if

$s=1/2\pm ir_n$, $n\in N_o$. Satz 5.6, Satz 5.7 of [Ro1] imply that

$$\mathscr{H}_k(s)=\{f\in\mathscr{D}_k \mid -\Delta_k f=s(1-s)f\}.$$

Denote by $d_{\Gamma,\chi}(s)$ the multiplicity of the eigen value $s(1-s)$ of the self-adjoint operator $-\Delta_k^{\sim}$. Obviously,

$$d_{\Gamma,\chi}(s)=\dim \mathscr{H}_k(s)\,.$$

We concentrate our attention to the case of $s=k$. The following lemma is due to Roelcke [Ro1, Satz 5.2] (cf. [Ch1, Hilfssatz 3]).

LEMMA 2.1. (i) *$\mathscr{H}_k(k)=\{f\in\mathscr{H}_k \mid y^{-k}f(z)$ is holomorphic on $\mathfrak{H}\}$.*

(ii) *The space $\mathscr{H}_k(k)$ is isomorphic to $Q(\Gamma,\,-2k,\,\chi)$ via the correspondence $f\to y^{-k}f(z)$.*

(iii) *If $k>0$, then, $\mathfrak{S}_{2k}(\Gamma,\chi)=Q(\Gamma,\,-2k,\,\chi)$. Accordingly,*

$$\dim \mathfrak{S}_{2k}(\Gamma,\chi)=d_{\Gamma,\chi}(k) \qquad \text{if} \quad k>0\,.$$

Proof. The assertion (i) is nothing but Satz 5.2 of [Ro1]. The assertion (ii) is obvious from (i). Take $F\in Q(\Gamma,\,-2k,\,\chi)$ and put $f(z)=y^kF(z)$. Then $f\in\mathscr{D}_k$, and f has a Fourier expansion at each cusp of Γ of the form as in [Fi, Proposition 1.5.4]. Since $f\in\mathscr{D}_k$, f is square integrable at every cusp of Γ in the sense of Definition 2.1 of [Ro1]. Therefore if $k>0$, the constant term of the Fourier expansion of f at every cusp of Γ vanishes. That means $F\in\mathfrak{S}_{2k}(\Gamma,\chi)$. The opposite inclusion $\mathfrak{S}_{2k}(\Gamma,\chi)\subset Q(\Gamma,\,-2k,\,\chi)$ is easy to see. q.e.d.

Now suppose that $k\geq 1/2$. We consider the problem of representing $d_{\Gamma,\chi}(k)$ (or $\dim \mathfrak{S}_{2k}(\Gamma,\chi)$) in another arithmetic or analytic quantities by using the resolvent trace formula and also Proposition 1.2.

We note here that the function $\xi_{hyp}(s)=(Z'_{\Gamma,\chi}/Z_{\Gamma,\chi})(s)$ can be continued analytically to a meromorphic function in the whole s-plane through the resolvent trace formula. We set

$$\mu(\Gamma,\chi;k)=i\Bigg(\sum_{\{R\}_\Gamma\, 0<\theta<\pi} \operatorname{tr}\chi(R)\cdot\frac{e^{i(2k-1)\theta}}{2\nu(R)\sin\theta}\Bigg)+\frac{d\tau}{2}-\sum_{j=1}^{\tau}\sum_{p=1}^{d}\beta_{jp}-\tau^*\,.$$

The following proposition is rather a direct consequence of Fischer's resolvent trace formula.

PROPOSITION 2.2. *Assume that $-1_2\in\Gamma$ and $1/2\leq k\leq 1$. Let χ be a unitary multiplier system of Γ of weight $2k$ and dimension d.*

(i) *If $k=1/2$, then,*

$$\dim \mathfrak{S}_1(\Gamma,\chi)=\frac{1}{2}\operatorname{Res}_{s=1/2}\left(\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s)\right)+\frac{1}{2}\mu(\Gamma,\chi;1/2)+\frac{1}{4}(\tau^*-\operatorname{tr}\Phi(1/2))\,.$$

(ii) *If $1/2<k\leq 1$, then,*

$$\dim \mathfrak{S}_{2k}(\Gamma, \chi) = \mathrm{Re}\, s_{s=k}\left(\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s)\right) + \frac{d \operatorname{vol}(\Gamma \backslash \mathfrak{H})}{4\pi}(2k-1) + \mu(\Gamma, \chi; k)\,.$$

In particular, if $k=1$, *then,*

$$\mathrm{Re}\, s_{s=1}\left(\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s)\right) = d_0(\Gamma, x)\,,$$

where

(2.1) $$d_0(\Gamma, \chi) = \dim\{v \in V \mid \chi(M)v = v \quad \textit{for all} \quad M \in \Gamma\}\,.$$

REMARK. In the case of $k=1$ the expression for $\dim \mathfrak{S}_2(\Gamma, \chi)$ in (ii) will be derived from the result of Ishikawa [Ish, Theorem].

Proof. Note that $\dim \mathfrak{S}_{2k}(\Gamma, \chi) = d_{\Gamma,\chi}(k)$ by Lemma 2.1. We choose a so that $\mathrm{Re}\, a > \max(1, k)$. First assume that $k=1/2$. According to the definition (1.18), $(2s-1)S_{\Gamma,\chi}(s, a)$ as a meromorphic function of s has a simple pole at $s=1/2$ with the residue $2d_{\Gamma,\chi}(1/2)$. We see from the expressions (1.11), (1.16) that $\xi_I(s)$ is holomorphic at $s=1/2$ and that $\xi_{ell}(s)$ has a simple pole at $s=1/2$ with the residue

$$\sum_{\{R\}_\Gamma\, 0<\theta<\pi} \frac{\operatorname{tr} \chi(R)}{2\nu(R)\sin\theta}\,.$$

By virtue of [Fi, (2.4.6)], the last integral on the right hand side of (1.17), as a function of s, has at most a simple pole at $s=1/2$. Therefore $\xi_{par}(s)$ has a simple pole at $s=1/2$ with the residue

$$\frac{d\tau}{2} - \sum_{j=1}^{\tau} \sum_{p=1}^{d} \beta_{jp} - \tau^* + \frac{1}{2}\operatorname{tr}(1_{\tau^*} - \Phi(1/2))\,.$$

Thus calculating the residues of $(2s-1)S_{\Gamma,\chi}(s, a)$ at $s=1/2$ in two manners with the help of the resolvent trace formula (1.19), we obtain the assertion (i). Next let $1/2 < k \le 1$. The function $(2s-1)S_{\Gamma,\chi}(s, a)$, a being fixed, has a simple pole at $s=k$ with the residue $d_{\Gamma,\chi}(k)$. Comparing the residues of the both sides of (1.19) at $s=k$ yields the expression for $d_{\Gamma,\chi}(k)$ in the assertion (ii). Finally let $k=1$. We note that the Selberg zeta function $Z_{\Gamma,\chi}(s)$ depends on k mod $\boldsymbol{Z}$. We also use the resolvent trace formula (1.19) for $k=0$ and for the same χ. Then by comparing the residues of the both sides of (1.19) at the simple pole $s=1$, we obtain, for $k=0$,

$$d_{\Gamma,\chi}(1) = \mathrm{Re}\, s_{s=1}\left(\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s)\right).$$

On the other hand, for $k=0$,

$$d_{\Gamma,\chi}(1) = \dim\{f \in \mathscr{D}_0 \mid -\Delta_0 f = 0\}\,.$$

Thanks to [Ro1, Satz 5.1], every eigen function $f \in \mathscr{D}_0$ of Δ_0 with the eigen value zero is a constant function. Therefore we easily get the identity (2.1) in the assertion (ii).

q.e.d.

We set

$$H(s, r)=\frac{\Gamma(s+ir)\Gamma(s-ir)}{\Gamma(2s)} \qquad (s, r \in \boldsymbol{C}).$$

We choose $H(s, r)$ as a test function $h(r)$ and study the both sides of the general Selberg trace formula (1.22) as meromorphic functions of s. By Stirling's formula for the gamma function, we have the estimate

$$\Gamma(s+ir)\Gamma(s-ir)=O(|r|^{2\mathrm{Re}\,s-1}e^{-\pi|r|}) \qquad \text{as} \quad |r|\to+\infty \tag{2.2}$$

with s lying in a compact set of the s-plane. For any fixed $r\in\boldsymbol{C}$, the function $H(s, r)$ is a meromorphic function in the whole s-plane which has only simple poles. Define a subset S of $\boldsymbol{C}$ by

$$S=\{\pm ir_n-m \mid m, n\in \boldsymbol{N}_0\}.$$

By [Fi, Theorem 1.6.5] and the estimate (2.2), the infinite series $\sum_{n=0}^{\infty} H(s, r_n)$ is absolutely and uniformly convergent with respect to s in any compact set contained in $\boldsymbol{C}-S$, and hence indicates a holomorphic function in $\boldsymbol{C}-S$. Since the poles of $H(s, r_n)$ are located at $s=\pm ir_n-m$ $(m\in\boldsymbol{N}_0)$ and the number of n with the same value r_n is finite, we see easily that

(2.3) $\sum_{n=0}^{\infty} H(s, r_n)$ is a meromorphic function in the whole s-plane having only simple poles.

Set, for $s\in\boldsymbol{C}$, $u\in\boldsymbol{R}$,

$$G(s, u)=\left(2\cosh\frac{u}{2}\right)^{-2s}.$$

Then it is easy to observe that the inverse Fourier transform of $G(s, u)$ coincides with $H(s, r)$:

$$H(s, r)=\int_{-\infty}^{\infty} G(s, u)e^{iru}du \qquad \text{(see [Ch2, §4])}.$$

The following proposition gives a generalization of [Ch2, Hilfssatz 27] and [Ta-Ish, Corollary to Theorem 1].

Proposition 2.3. *The function $\zeta_{\Gamma,\chi}(s)$ given in (1.15) can be continued analytically to a meromorphic function in the whole s-plane having only simple poles.*

Proof. We need a lemma due to Christian [Ch2, Hilfssatz 22] (see also [Ta-Ish, p. 127]).

Lemma 2.4. *Let $f(\zeta)$ be a meromorphic function in the whole complex plane* *which is holomorphic on the real line* $\mathbf{R}$. *Assume that, for some* ε *with* $0<\varepsilon<\pi$,

$$\int_{-\infty}^{\infty} e^{-(\pi-\varepsilon)|r|}|f(r)|dr<+\infty .$$

Set, for $\operatorname{Re} s>0$,

$$J(f, s)=\frac{1}{2\pi}\int_{-\infty}^{\infty} H(s, r)f(r)dr ,$$

which is absolutely convergent by the estimate (2.2). *Then,* $J(f, s)$ *entails the following properties*:

(i) $J(f, s)$ *is continued analytically to a meromorphic function in the whole s-plane.*

(ii) $J(f, s)$ *is holomorphic in the half plane* $\operatorname{Re} s\geq 0$.

(iii) *If* $f(\zeta)$ *has only simple poles, so does* $J(f, s)$.

The proof is quite similar to that of [Ch2, Hilfssatz 22], so we omit it.

We continue the proof of Proposition 2.3. We take $H(s, r)$ as a test function $h(r)$. First we choose $\operatorname{Re} s$ sufficiently large (for instance $\operatorname{Re} s>k^*+\delta$). Then, s being fixed, $h(r)=H(s, r)$ satisfies the condition (1.20) and hence Proposition 1.2 can be applied to this special choice of $h(r)$. Thus the identity (1.22) holds for $h(r)=H(s, r)$ if $\operatorname{Re} s$ is sufficiently large. By the assertion (2.3), the left side of (1.22) is a meromorphic function of s in $\boldsymbol{C}$ having only simple poles. The function $f_1(\zeta)$ in (1.23) satisfies the condition of Lemma 2.4. Thanks to Lemma 2.4.19, Corollary 2.4.17 of [Fi], the function $(\varphi'/\varphi)(1/2+i\zeta)$, as a function of ζ, also satisfies the condition of Lemma 2.4. Moreover we have

$$g(u)=G(s, u)=\left(2\cosh\frac{u}{2}\right)^{-2s} .$$

In view of Lemma 2.4 we observe that by virtue of the identity (1.22) the function $\zeta_{\Gamma,\chi}(s)$ can be analytically continued to a meromorphic function in the whole s-plane which has only simple poles. q.e.d.

THEOREM 2.5. *Assume that* $-1_2\in\Gamma$. *Let* $1/2\leq k\leq 1$ *and let* χ *be a unitary multiplier system of* Γ *of weight* $2k$ *and dimension* d.

(i) *If* $k=1/2$, *then,*

$$\dim \mathfrak{S}_1(\Gamma, \chi)=\frac{1}{2}\operatorname{Res}_{s=0}\zeta_{\Gamma,\chi}(s)+\frac{1}{2}\mu(\Gamma, \chi;\, 1/2)+\frac{1}{4}(\tau^*-\operatorname{tr}\Phi(1/2)) .$$

(ii) *If* $1/2<k\leq 1$, *then,*

$$\dim \mathfrak{S}_{2k}(\Gamma, \chi)=\operatorname{Re} s_{s=k-1/2}\zeta_{\Gamma,\chi}(s)+\frac{d\operatorname{vol}(\Gamma\backslash\mathfrak{H})}{4\pi}(2k-1)+\mu(\Gamma, \chi;\, k) .$$

REMARK. Our result, (i) of Theorem 2.5 is consistent with Theorem F of [Hi2], where $\tau=1$ and χ is a character of Γ with $\chi(-1_2)=-1$.

Proof. We note that the infinite series $\sum_{n=0}^{\infty} H(s, r_n)$, as a meromorphic function of s, has a simple pole at $s=k-1/2$ with the residue $d_{\Gamma,\chi}(k)$ (resp. $2d_{\Gamma,\chi}(k)$) if $k>1/2$ (resp. $k=1/2$). We set $h(r)=H(s, r)$ in the identity (1.22). Then we observe the both sides of the identity (1.22) as meromorphic functions of s and compare the residues at the simple pole $s=k-1/2$ of the both sides. Thus we get the assertions (i), (ii).

q.e.d.

Comparing the dimension formulas in Proposition 2.2 and Theorem 2.5, we immedaitely have

COROLLARY 2.6. *Let the notation be the same as in Theorem* 2.5. *If* $1/2\le k\le 1$, *then,*

$$\mathrm{Res}_{s=k}\left(\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s)\right)=\mathrm{Res}_{s=k-1/2}\zeta_{\Gamma,\chi}(s)\,.$$

In particular if $k=1$, *then this value is equal to* $d_0(\Gamma, \chi)$ *in* (2.1).

Now we explain some applications of Proposition 2.2 and Theorem 2.5. Assume that χ is a character of Γ with $\chi(-1_2)=-1$. Then, χ forms a unitary multiplier system of Γ of weight 1 with $d=1$. Choose $\zeta_1, \cdots, \zeta_\tau; A_1, \cdots, A_\tau\in SL_2(\boldsymbol{R})$; $T_j=A_j^{-1}UA_j$ $(1\le j\le\tau)$ in the same manner as in §1. We may take $v_{j1}=1$ for each j and write, instead of (1.3),

$$\chi(T_j)=\exp(2\pi i\beta_j)\,,\qquad \text{where}\quad \begin{pmatrix}\beta_j=0 & \cdots & \text{if}\ \ m_j=1\,,\\ 0<\beta_j<1 & \cdots & \text{if}\ \ m_j=0\,.\end{pmatrix} \tag{2.4}$$

Denote by I_χ the set of indices j $(1\le j\le\tau)$ with $\beta_j=0$. Then,

$$\tau^*=\sharp(I_\chi)\,, \tag{2.5}$$

where $\sharp(E)$ stands for the cardinality of a set E. For any $j, l\in I_\chi$, we write simply $\varphi_{jl}(s)$ for $\varphi_{j1,l1}(s)$. It follows from Lemma 1.1 that

$$\varphi_{jl}(s)=e^{-\pi i/2}\gamma(s, 1/2)\sum_{M\in\Gamma^+_{\zeta_j}\backslash B^+_{jl}(\Gamma)/\Gamma^+_{\zeta_l}}\frac{\chi(M)^{-1}}{c(A_jMA_l^{-1})^{2s}}\,. \tag{2.6}$$

THEOREM 2.7. *Assume that* $-1_2\in\Gamma$. *Let* χ *be a* $\{\pm1\}$*-valued character of* Γ *with* $\chi(-1_2)=-1$. *Then,*

$$\dim\mathfrak{S}_1(\Gamma, \chi)=\frac{1}{2}\,\mathrm{Re}\ s_{s=0}\zeta_{\Gamma,\chi}(s)=\frac{1}{2}\,\mathrm{Re}\ s_{s=1/2}\left(\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s)\right).$$

Proof. Since χ is $\{\pm1\}$-valued, the value β_j in (2.4) takes the value 0 or 1/2 according as $m_j=1$ or 0. Thus,

$$\sum_{j=1}^{\tau} \beta_j + \frac{\tau^*}{2} = \frac{\tau}{2} .$$

The identity (2.6) implies that

$$\overline{\varphi_{jl}(\bar{s})} = -\varphi_{jl}(s) \qquad \text{for} \quad j, l \in I_\chi .$$

Hence it is easy to see from (1.8) that $\operatorname{tr} \Phi(s) = 0$. Since χ is $\{\pm 1\}$-valued, the contribution from the elliptic conjugacy classes of Γ has to vanish. Thus we obtain, as an immediate consequence of Theorem 2.5, (i) and Corollary 2.6, the assertion of Theorem 2.7. q.e.d.

Another application of Proposition 2.2 and Theorem 2.5 is Theorem 0.2 given in the introduction. Here we give a proof.

Proof of Theorem 0.2. Let $\zeta_1, \cdots, \zeta_\tau; A_1, \cdots, A_\tau \in SL_2(\boldsymbol{R})$; $T_j = A_j^{-1} U A_j$ $(1 \leq j \leq \tau)$ be the same as before with respect to the group $\Gamma = \Gamma_0(N)$. Let χ be a character of $\Gamma_0(N)$ given by (0.1) in the introduction. Set

$$\omega_N = N^{-1/2} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in SL_2(\boldsymbol{R}) .$$

The element ω_N normalizes the group $\Gamma_0(N)$:$\omega_N \Gamma_0(N) \omega_N^{-1} = \Gamma_0(N)$. Then,

$$\chi(\omega_N M \omega_N^{-1}) = \chi(M)^{-1} \qquad \text{for any} \quad M \in \Gamma_0(N) . \tag{2.7}$$

For each j $(1 \leq j \leq \tau)$, $\omega_N \zeta_j$ is also a cusp of $\Gamma_0(N)$. Assume that $\omega_N \zeta_j$ is $\Gamma_0(N)$-equivalent to some cusp ζ_l $(1 \leq l \leq \tau)$ and write

$$\omega_N \zeta_j = \sigma \zeta_l \qquad \text{with some} \quad \sigma \in \Gamma_0(N) . \tag{2.8}$$

Then the stabilizer Γ_{ζ_l} and the generator $T_l = A_l^{-1} U A_l$ of $\Gamma_{\zeta_l}^+$ have the expressions:

$$\Gamma_{\zeta_l} = \sigma^{-1} \omega_N \Gamma_{\zeta_j} \omega_N^{-1} \sigma , \qquad T_l = \sigma^{-1} \omega_N T_j \omega_N^{-1} \sigma . \tag{2.9}$$

It follows from (2.7) that $\chi(T_l) = \chi(T_j)^{-1}$. Thus if $0 < \beta_j < 1$, then, $\beta_l = 1 - \beta_j$. For each $j \in I_\chi$, there exists a unique $l \in I_\chi$ with (2.8). Therefore we have, with the help of (2.5),

$$2 \sum_{j=1}^{\tau} \beta_j + \tau^* = \tau . \tag{2.10}$$

We set, for each $M \in \Gamma_{\zeta_j}$,

$$M^* = \sigma^{-1} \omega_N M \omega_N^{-1} \sigma .$$

Putting $g = A_j \omega_N^{-1} \sigma A_l^{-1}$, we get, by (2.9), $g^{-1} U g = U$. Therefore, g has a form $\pm \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ $(x \in \boldsymbol{R})$. Thus the relation $A_l M^* A_l^{-1} = g^{-1} A_j M A_j^{-1} g$ implies that

$$c(A_l M^* A_l^{-1}) = c(A_j M A_j^{-1}) . \tag{2.11}$$

We see from this fact that M runs over a complete set of representatives of the double cosets $\Gamma^+_{\zeta_j}\backslash B^+_{jj}(\Gamma)/\Gamma^+_{\zeta_j}$ if and only if M^* runs over that of $\Gamma^+_{\zeta_l}\backslash B^+_{ll}(\Gamma)/\Gamma^+_{\zeta_l}$. Moreover we have $\chi(M^*)=\chi(M)^{-1}$. Therefore the expression (2.6) shows us with the use of (2.11) that

$$\overline{\varphi_{jj}(\bar{s})}=-\varphi_{ll}(s)\,.$$

Taking the property (1.8) into account, we have

$$\varphi_{jj}(s)+\varphi_{ll}(s)=0\,.$$

When j runs over all elements of I_χ, so does the corresponding l. Therefore,

$$\operatorname{tr}\Phi(s)=0\,. \tag{2.12}$$

Let $R_1,\cdots,R_\rho$ be a complete system of representatives of the $\Gamma_0(N)$-equivalence classes of primitive elliptic elements and let $2\nu_1,\cdots,2\nu_\rho$ denote the orders of $R_1,\cdots,R_\rho$ (for the definition of primitive elliptic elements of Γ, see §1 or [Fi, Notation 2.3.3]). The contribution of the elliptic conjugacy classes of $\Gamma_0(N)$ to $\dim\mathfrak{S}_1(\Gamma_0(N),\chi)$ equals

$$\frac{i}{2}\sum_{\{R\}_\Gamma\,0<\theta<\pi}\frac{\chi(R)}{2\nu(R)\sin\theta}=\frac{i}{2}\sum_{j=1}^{\rho}\sum_{m=1}^{\nu_j-1}\frac{\chi(R_j)^m}{2\nu_j\sin(\pi m/\nu_j)}\,. \tag{2.13}$$

For each j $(1\le j\le\tau)$, the element $\omega_N R_j\omega_N^{-1}$ is also a primitive elliptic element of $\Gamma_0(N)$, and hence is $\Gamma_0(N)$-conjugate to some R_l. The corresponding l is uniquely determined by j, and $\nu_l=\nu_j$, $R_l^{\nu_l}=-1_2$. Since we have, by (2.7), $\chi(R_j)=\chi(R_l)^{-1}$, we observe that

$$\frac{\chi(R_j)^m}{2\nu_j\sin(\pi m/\nu_j)}=-\frac{\chi(R_l)^{\nu_l-m}}{2\nu_l\sin(\pi(\nu_l-m)/\nu_l)}\,.$$

Therefore the elliptic contribution (2.13) vanishes. Finally with the help of (2.10), (2.12), the assertion of Theorem 0.2 is reduced to the dimension formulas for $\dim\mathfrak{S}_1(\Gamma_0(N),\chi)$ in Proposition 2.2 and Theorem 2.5. q.e.d.

§3. The cases of $-1_2\notin\Gamma$

In the present paragraph we assume that Γ is a discrete subgroup of $SL_2(\boldsymbol{R})$ with $\mathrm{vol}(\Gamma\backslash\mathfrak{H})<+\infty$ not containing the element -1_2. Let $(V,\langle\,,\rangle)$, $\mathcal{U}(V)$ be the same as in §1. Let k be a real number. In this setting a map $\chi\colon\Gamma\to\mathcal{U}(V)$ is called a unitary multiplier system of Γ of weight $2k$ and dimension d $(d=\dim V)$, if χ satisfies

$$\chi(ST)=\sigma_{2k}(S,T)\chi(S)\chi(T)\qquad\text{for all}\quad S,T\in\Gamma\,.$$

We set

$$\Gamma^\sim=\Gamma\cup\{-\Gamma\}\,,\qquad\text{with }-\Gamma\text{ denoting the set }\{-M\mid M\in\Gamma\}\,. \tag{3.1}$$

Then, $\Gamma^\sim$ is a discrete subgroup of $SL_2(\boldsymbol{R})$ with $\mathrm{vol}(\Gamma^\sim\backslash\mathfrak{H})<+\infty$ and $-1_2\in\Gamma^\sim$. We define a map $\chi^\sim\colon\Gamma^\sim\to\mathcal{U}(V)$ by putting

$$(3.2)\qquad \chi^{\sim}(M)=\begin{cases}\chi(M) & \cdots \text{ if } M\in\Gamma\,,\\ \sigma_{2k}(-M,\,-1_2)e^{-2\pi ik}\chi(-M) & \cdots \text{ if } M\in-\Gamma\,.\end{cases}$$

LEMMA 3.1. *$\chi^{\sim}$ forms a unitary multiplier system of $\Gamma^{\sim}$ of weight $2k$ in the sense of* (1.1).

Proof. We need a lemma concerning factor systems $\sigma_{2k}(A, B)$.

LEMMA 3.2. $\sigma_{2k}(-A,\,-B)\sigma_{2k}(A,\,-1_2)\sigma_{2k}(B,\,-1_2)=e^{4\pi ik}\sigma_{2k}(A, B)$ *for any* A, $B\in SL_2(\boldsymbol{R})$.

Proof. Making use of the relations (1.3.3), (1.3.4), and (1.3.5) of [Fi], we have

$$\begin{aligned}\sigma_{2k}(-A,\,-B)\sigma_{2k}(A,\,-1_2)\sigma_{2k}(B,\,-1_2)&=\sigma_{2k}(A, B)\sigma_{2k}(-1_2,\,-B)\sigma_{2k}(-1_2,\,B)\\ &=\sigma_{2k}(A, B)\sigma_{2k}(1_2,\,B)\sigma_{2k}(-1_2,\,-1_2)=e^{4\pi ik}\sigma_{2k}(A, B)\,.\end{aligned}$$

q.e.d.

Now let S, $T\in\Gamma$. It is easy to see from Lemma 3.2 that

$$\chi^{\sim}((-S)(-T))=\sigma_{2k}(-S,\,-T)\chi^{\sim}(-S)\chi^{\sim}(-T)\,.$$

We get, again by (1.3.3), (1.3.5) of [Fi],

$$\begin{aligned}\chi^{\sim}((-S)T)&=\sigma_{2k}(ST,\,-1_2)e^{-2\pi ik}\chi(ST)\\ &=\sigma_{2k}(S,\,-T)\sigma_{2k}(T,\,-1_2)e^{-2\pi ik}\chi(S)\chi(T)\\ &=\sigma_{2k}(-S,\,T)\sigma_{2k}(S,\,-1_2)e^{-2\pi ik}\chi(S)\chi(T)\\ &=\sigma_{2k}(-S,\,T)\chi^{\sim}(-S)\chi^{\sim}(T)\,.\end{aligned}$$

Similarly, we have $\chi^{\sim}(S(-T))=\sigma_{2k}(S,\,-T)\chi^{\sim}(S)\chi^{\sim}(-T))$. The relation $\chi^{\sim}(-1_2)=e^{-2\pi ik}\mathrm{id}_V$ is immediate from (3.2). q.e.d.

We now define the Selberg zeta function $Z_{\Gamma,\chi}(s)$ and the function $\zeta_{\Gamma,\chi}(s)$ associated with Γ, χ via the corresponding $Z_{\Gamma^{\sim},\chi^{\sim}}(s)$, $\zeta_{\Gamma^{\sim},\chi^{\sim}}(s)$, which are given by (1.13), (1.15). We set, for any hyperbolic element P of Γ,

$$\chi^{*}(P)=\begin{cases}\chi(P) & \cdots \ \operatorname{tr} P>2\\ \sigma_{2k}(P,\,-1_2)e^{-2\pi ik}\chi(P) & \cdots \ \operatorname{tr} P<-2\,.\end{cases}$$

We note here that

$$\chi^{*}(P)=\operatorname{sgn}(\operatorname{tr} P)\chi(P)\qquad \text{if}\quad k\equiv 1/2 \bmod \boldsymbol{Z}\,.$$

Also in the case of $-1_2\notin\Gamma$, every hyperbolic element P of Γ has an expression of the form (1.12). Denote by $\{P\}_\Gamma$ (resp. $\{P_0\}_\Gamma$) the Γ-conjugacy classes of hyperbolic (resp. primitive hyperbolic) elements of Γ. The Selberg zeta function $Z_{\Gamma,\chi}(s)$ is defined by

$$(3.3)\qquad Z_{\Gamma,\chi}(s)=\prod_{\{P_0\}_\Gamma}\prod_{m=0}^{\infty}\det(\mathrm{id}_V-\chi^*(P_0)N(P_0)^{-s-m}).$$

We also define the zeta function $\zeta_{\Gamma,\chi}(s)$ as follows:

$$\zeta_{\Gamma,\chi}(s)=\sum_{\{P\}_\Gamma}\frac{\mathrm{tr}\,\chi^*(P)\log N(P_0)}{N(P)^{1/2}-N(P)^{-1/2}}\left(2\cosh\frac{\log N(P)}{2}\right)^{-2s},$$

where P_0 is the primitive hyperbolic element associated to P.

PROPOSITION 3.3. *$Z_{\Gamma,\chi}(s)=Z_{\Gamma^\sim,\chi^\sim}(s)$ and $\zeta_{\Gamma,\chi}(s)=\zeta_{\Gamma^\sim,\chi^\sim}(s)$.*

The proof is immediate from the definition of χ^*, $\chi^\sim$.

Using the corresponding results for $Z_{\Gamma^\sim,\chi^\sim}(s)$, $\zeta_{\Gamma^\sim,\chi^\sim}(s)$, we see from Proposition 3.3 that $Z_{\Gamma,\chi}(s)$ (resp. $\zeta_{\Gamma,\chi}(s)$) is absolutely convergent for $\mathrm{Re}\,s>1$ (resp. $\mathrm{Re}\,s>1/2$) and that $(Z'_{\Gamma,\chi}/Z_{\Gamma,\chi})(s)$, $\zeta_{\Gamma,\chi}(s)$ have meromorphic continuations to the whole s-plane.

Denote by $\mathfrak{S}_{2k}(\Gamma,\chi)$ the space of cusp forms of weight $2k$ with respect to Γ, χ which is similarly defined as in the case of $-1_2\in\Gamma$. Since $\chi^\sim(-M)J(-M,z)^{2k}=\chi(M)J(M,z)^{2k}$ for any $M\in\Gamma$, each $f\in\mathfrak{S}_{2k}(\Gamma,\chi)$ is naturally identified with an element of $\mathfrak{S}_{2k}(\Gamma^\sim,\chi^\sim)$. Thus the space $\mathfrak{S}_{2k}(\Gamma,\chi)$ is isomorphic to $\mathfrak{S}_{2k}(\Gamma^\sim,\chi^\sim)$ via the correspondence $f\to f$. Therefore one can derive some formulas for $\dim\mathfrak{S}_{2k}(\Gamma,\chi)$ using the results of §2. One of such formulas is the following.

THEOREM 3.4. *Assume that $-1_2\notin\Gamma$. Let χ be a $\{\pm1\}$-valued character of Γ. Then,*

$$\dim\mathfrak{S}_1(\Gamma,\chi)=\frac{1}{2}\mathrm{Res}_{s=0}\zeta_{\Gamma,\chi}(s)=\frac{1}{2}\mathrm{Res}_{s=1/2}\left(\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s)\right).$$

Proof. Let $\Gamma^\sim$, $\chi^\sim$ be the same as in (3.1), (3.2). Then, $\chi^\sim(M)=\chi(M)$ and $\chi^\sim(-M)=-\chi(M)$ for $M\in\Gamma$. Furthermore, $\chi^\sim$ forms a $\{\pm1\}$-valued character of Γ with the condition $\chi^\sim(-1_2)=-1$. Therefore the assertion follows immediately from Theorem 2.7, Proposition 3.3, since $\dim\mathfrak{S}_1(\Gamma,\chi)=\dim\mathfrak{S}_1(\Gamma^\sim,\chi^\sim)$. q.e.d.

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